Exc 1

$$A = \begin{bmatrix} -1 & 3 & 0 & 0 \\ -1 & -2 & -2 & 0 \\ 0 & 4 & -3 & 6 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

a) Graph : ① ⇄ ② ⇄ ③ ⇄ ④

Graph is strongly connected, since you can get from any node / unknown to any other node / unknown

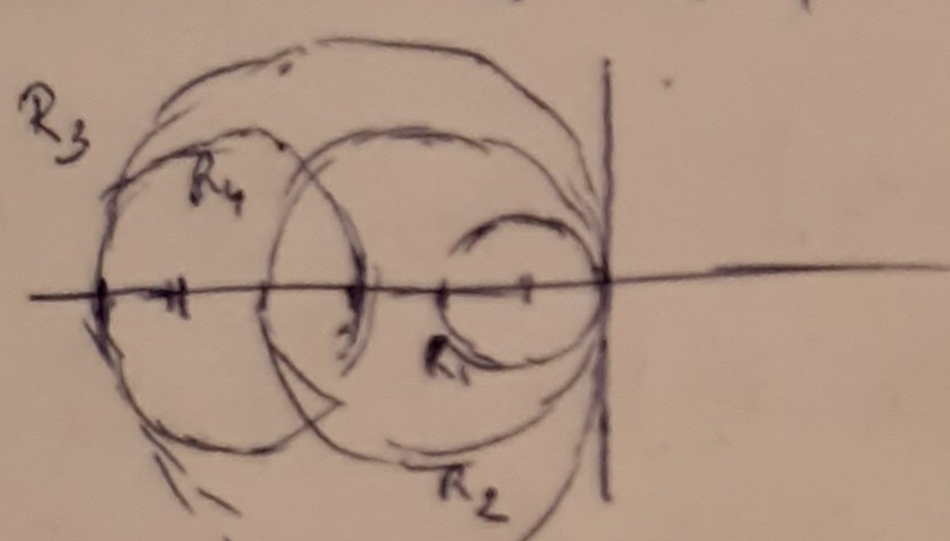
$\Rightarrow A$ irreducible

b) $A = A_1 + A_2$ $\quad A_1 = \frac{A+A^T}{2}$ symmetric, $A_2 = \frac{A-A^T}{2}$ skew-symmetric

$$A_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

A_1 symmetric $\Rightarrow$ Gerschgorin applied to columns will not give additional information

$R_1 = \{z \mid |z+1| < 1\}$ $R_2 = \{z \mid |z+2| < 2\}$ $R_3 = \{z \mid |z+3| < 3\}$ $R_4 = \{z \mid |z+4| < 2\}$



GGT : $\sigma(A_1) \subset \bigcup_{i=1}^{4} R_i = R_3$

A_1 irreducible : λ of A_1 on boundary of $\bigcup_{i=1}^{4} R_i$ only if λ on boundary of all R_i

A_1 symmetric $\Rightarrow \lambda$ real $\Rightarrow \lambda \in [-6, 0]$

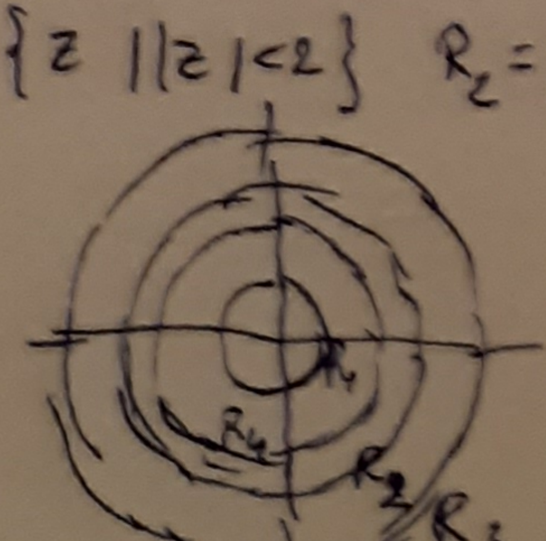
but -6 and 0 not on boundary of each R_i

$\Rightarrow \sigma(A_1) \in (-6, 0)$

$$A_2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -3 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & -4 & 0 \end{bmatrix}$$

A_2 skew-symm $\Rightarrow$ Gerschgorin applied to columns will not give additional information

$R_1 = \{z \mid |z| < 2\}$ $R_2 = \{z \mid |z| < 5\}$ $R_3 = \{z \mid |z| < 7\}$ $R_4 = \{z \mid |z| < 4\}$



GGT : $\sigma(A_2) \subset \bigcup_{i=1}^{4} R_i = R_3$

A_2 skew symm $\Rightarrow \lambda$ purely imaginary $\Rightarrow \lambda \in [-7i, 7i]$

A_2 irreducible : λ of A_2 on boundary of $\bigcup_{i=1}^{4} R_i$ only if λ on boundary of all R_i

but $-7i$ and $7i$ not on boundary of each R_i

$\Rightarrow \sigma(A_2) \subset (-7i, 7i)$

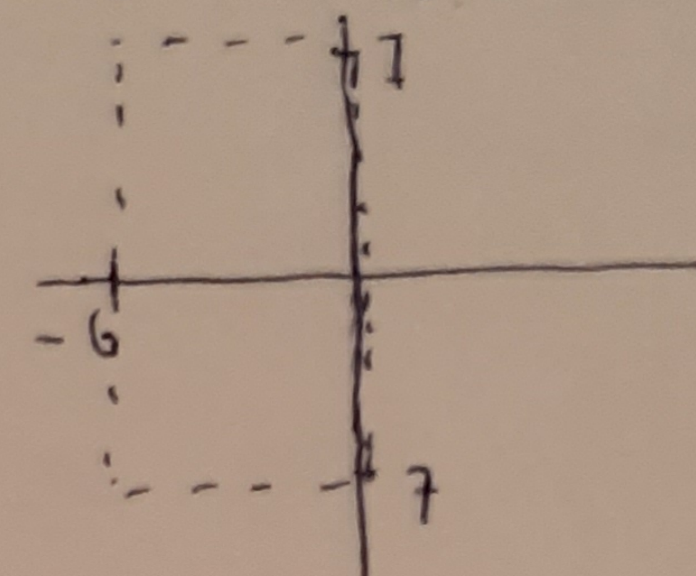
c Bendixson's theorem $F(A) \subset F\left(\frac{A+A^T}{2}\right) + F\left(\frac{A-A^T}{2}\right)$

furthermore: $\sigma(A) \subset F(A)$

$\left.\begin{array}{l} A_1 = \frac{A+A^T}{2} \\ A_2 = \frac{A-A^T}{2} \end{array}\right\}$ both normal $\Rightarrow$ $F(A_1)$ and $F(A_2)$ convex hull of $\sigma(A_1)$ and $\sigma(A_2)$ respectively

$\sigma(A) \subset F(A) \subset F(A_1) + F(A_2) = \{ z \in \mathbb{C} \mid -6 < Re(z) < 0,\ -7 < Im(z) < 7 \}$

$\Rightarrow$ $\sigma(A)$ in box with boundaries not included



hence $\lambda = 0 \notin \sigma(A)$ $\Rightarrow$ A non-singular

Exc 2

a $y^{(k)} = A^k X$ $\quad X = [x_1, x_2]$ $\quad x_1 = \sum_{i=1}^n v_i$ $\quad x_2 = \sum_{i=1}^n \frac{1}{i} v_i$

$\Rightarrow y^{(k)} = A^k X = A^k [x_1, x_2] = [A^k x_1, A^k x_2]$

$= \left[\sum_{i=1}^n A^k v_i, \sum_{i=1}^n \frac{1}{i} A^k v_i \right] = \left[\sum_{i=1}^n \lambda_i^k v_i, \sum_{i=1}^n \frac{1}{i} \lambda_i^k v_i \right]$

$= \left[\lambda_1^k v_1 + \lambda_2^k v_2 + \sum_{i=3}^n \lambda_i^k v_i,\ \lambda_1^k v_1 + \frac{1}{2} \lambda_2^k v_2 + \sum_{i=3}^n \frac{1}{i} \lambda_i^k v_i \right]$

since $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq \ldots \geq |\lambda_n|$

λ_1^k and λ_2^k are larger than λ_i^k $\quad i = 3, 4 \ldots$

$\Rightarrow y^{(k)} \rightarrow [\lambda_1^k v_1 + \lambda_2^k v_2,\ \lambda_1^k v_1 + \frac{1}{2} \lambda_2^k v_2]$ for $k \to \infty$

b consider $(y^{(k)})^T A y^{(k)} \hat{y} = \Theta (y^{(k)})^T y^{(k)} \hat{y}$

for $\hat{y}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$: $y^{(k)} \hat{y}_1 = [\lambda_1^k v_1 + \lambda_2^k v_2 \quad \lambda_1^k v_1 + \frac{1}{2} \lambda_2^k v_2] \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$= \lambda_1^k v_1 + \lambda_2^k v_2 - 2\lambda_1^k v_1 - \lambda_2^k v_2 = -\lambda_1^k v_1$

$\hat{y}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$: $y^{(k)} \hat{y}_2 = \lambda_1^k v_1 + \lambda_2^k v_2 - (\lambda_1^k v_1 + \frac{1}{2} \lambda_2^k v_2)$

$= \frac{1}{2} \lambda_2^k v_2$

for large k

Hence $(Y^{(k)})^T A Y^{(k)} \hat{y}_1 = \theta (Y^{(k)})^T Y^{(k)} \hat{y}_1$

$\Leftrightarrow (Y^{(k)})^T A (-\lambda_1^k v_1) = \theta (Y^{(k)})^T (-\lambda_1^k v_1)$

$\Leftrightarrow -\lambda_1^k (Y^{(k)})^T (A v_1 - \theta v_1) = 0$

this holds if $\theta = \lambda_1$ since v_1 eigenvector of A

Similarly: $(Y^{(k)})^T A Y^{(k)} \hat{y}_2 = \theta (Y^{(k)})^T Y^{(k)} \hat{y}_2$

$\Leftrightarrow (Y^{(k)})^T A (\frac{1}{2}\lambda_2^k v_2) = \theta (Y^{(k)})^T (\frac{1}{2}\lambda_2^k v_2)$

$\Leftrightarrow \frac{1}{2}\lambda_2^k (Y^{(k)})^T (A v_2 - \theta v_2) = 0$

this holds if $\theta = \lambda_2$

Premultiplication by $(Y^{(k)})^T$ does not create more solutions since $Y^{(k)}$ is rank 2. Hence $\theta_1 = \lambda_1$, $\theta_2 = \lambda_2$ are the solutions

c Due to round-off and the fact that $|\lambda_2| < |\lambda_1|$ the contribution of v_2 in $Y^{(k)}$ will be eventually lost in rounding ($\lambda_1^k v_1$ much larger than $\lambda_2^k v_2$)

So in fact $Y^{(k)}$ will become rank 1

Hence $Y^{(k)T} Y^{(k)}$ will become singular making (1) unsolvable

The remedy is to reorthogonalize frequently. So

$X^{(0)} = X$

for $i = 1, \ldots$

$Y^{(k)} = A X^{(i-1)}$

orthogonalize $Y^{(i)} \to X^{(i)}$

end

Exc 3

a. $K^m(A, v) = \text{span}\{v, Av, \ldots, A^{m-1}v\}$

b. assume q_i, $i = 1 \ldots k$ are eigenvectors of A belonging to $\lambda_1, \ldots, \lambda_k$, the distinct eigenvalues

- then we can write $v = \sum_{i=1}^{k} \alpha_i q_i$ (q_i in invariant subspace associated with λ_i)

$\Rightarrow v \in \text{span}\{q_1, \ldots, q_k\}$

- also $A^\ell v = \sum_{i=1}^{k} \alpha_i A^\ell q_i = \sum_{i=1}^{k} \alpha_i \lambda_i^\ell q_i \in \text{span}\{q_1, \ldots, q_k\}$, $\ell = 1, 2, \ldots$

$\Rightarrow y \in \mathcal{K}^m(A, v) = \text{span}\{v, Av, \ldots, A^{m-1}v\}$ arbitrary elt.

$\Rightarrow y \in \text{span}\{q_1, \ldots, q_k\} \Rightarrow \dim \mathcal{K}^m(A, v) \leq k$

Alternative proof:

$v = \sum_{i=1}^{n} \beta_i v_i$ with v_i $i = 1, \ldots, n$ complete set of eigenvectors, i.e basis for $\mathbb{R}^n$

$A^\ell v = \sum_{i=1}^{n} \beta_i A^\ell v_i = \sum_{i=1}^{n} \beta_i \lambda_i^\ell v_i$ $\quad \ell = 0, 1, 2, \ldots$

however, there are equal eigenvalues

so $A^\ell v = \sum_{i=1}^{n} \beta_i \lambda_i^\ell v_i = \sum_{i=1}^{k} \lambda_i^\ell \underbrace{\sum_{j=1}^{r_i} \beta_{ij} v_j}$ $= \sum_{i=1}^{k} \lambda_i^\ell q_i$

is some vector, say q_i, in the invariant subspace associated with λ_i

$\in \text{span}\{q_1, \ldots, q_k\}$

$\Rightarrow y \in \mathcal{K}^m(A, v)$ arbitrary elt $\quad \mathcal{K}^m(A, v) = \text{span}\{v, Av, \ldots, A^{m-1}v\}$

$\Rightarrow y \in \text{span}\{q_1, \ldots, q_k\}$